Interaction Analysis in Multivariable Control Systems

Most published "interaction measures" can produce misleading results and often do not measure the true closed-loop interaction. Therefore, closed-loop transmittances are defined mathematically as *direct, parallel, disturbance,* and *interaction* transfer functions and it is shown that the direct Nyquist array design method includes an excellent basis for interaction analysis.

N. Jensen, D. G. Fisher and S. L. Shah
Department of Chemical Engineering
University of Alberta
Edmonton, Alberta, Canada T6G 2G6

SCOPE

Multivariable processes are typically much more difficult to design and operate than single-input/single-output processes, due to the interactions that occur between the input/output variables. For example, changing one input variable, r_i, on a process system may cause changes in several output variables, y_j, and hence make it difficult to maintain product quality. Similarly, the performance of one feedback control loop can be strongly affected by the controller parameters used in other loops on the same multivariable system. Therefore, in the past few years a significant amount of research has been done on the analysis of process interactions.

Interaction analysis is strongly affected by the amount of information that is available, i.e.,

1. Steady state model of the process
2. Dynamic model of the process
3. Control strategy, e.g., pairing of variables for feedback
4. Controller design and parameter values

If all of the above information is available then the process interactions can be calculated exactly and the closed-loop behavior of the process can be simulated. However, this is not of much help during the initial stages of designing a multiloop controller because 3 and 4 are not available. The relative gain array (RGA) proposed by Bristol (1966) would appear to be ideal because it can be calculated using only the steady state process gains and provides guidelines for pairing input/output variables and for predicting when interactions are significant. The RGA method is useful in many applications and is a topic in many textbooks. Unfortunately, it is easy to show by example that the RGA frequently fails to indicate when a system has significant interaction problems and may even give misleading information about steady state interactions. Also, there is rarely any indication that the RGA has failed until the closed-loop process is simulated or operated. Difficulties such as these led to the development of alternative interaction measures by Rijnsdorp (1965), Bristol (1968, 1977, 1978), Suchanti and Fournier (1973), Witcher and McAvoy (1977), Tung and Edgar (1977, 1978, 1981), McAvoy (1979, 1981, 1983a, b), Gagnepain and Seborg (1979), Jaaksoo (1979), and others.

This paper includes a critical review of these interaction measures and shows that the various authors have not used a consistent definition of interaction and that the proposed dynamic interaction measures usually do not estimate the actual interaction in the *closed-loop* system.

This paper therefore develops a formal definition of interaction and divides the transmittances between the input/output variables of a process into *direct, parallel,* and *interaction* components. Also, since most of the published interaction measures have significant shortcomings, it is recommended that when a dynamic model of the *open-loop* system is available (point 2 above), the various transmittances and interactions in the closed-loop system be evaluated by examining the process transfer function matrix and/or the corresponding direct Nyquist array (DNA) plots.

CONCLUSIONS AND SIGNIFICANCE

It is shown that the closed-loop transfer functions relating the process output variables, y, to the reference (setpoint) inputs, r, and the disturbance inputs, ξ, can be expressed mathematically as the sum of four transmittances:

1. Direct ($r_i \rightarrow y_i$)
2. Parallel ($r_i \rightarrow y_i$ via other loops)
3. Interaction ($r_i \rightarrow y_i$ where $j = 1, 2 \ldots m, j \neq i$)
4. Disturbance ($\xi_j \rightarrow y_i$ where $j = 1, 2, 3 \ldots m$)

It is obvious that in some applications the *parallel* transmittance can be very important relative to the *direct* transmittances and hence that performance will suffer if this transmittance is minimized via controller design. As expected, in the limit as the feedback gains approach infinity the direct *plus* parallel transmittance approaches unity while the interaction transmittance and the disturbance transmittance approach zero.

The critical review of published interaction measures showed that the interaction quotient, relative gain array, relative dynamic gain array, average gain array, and transient response functions all have the following shortcomings:

• The analysis applies to a system with one loop open (and $n - 1$ loops closed) rather than to a fully closed loop system.

• They measure the *parallel* transmittance (with one loop open) rather than the true *interaction* transmittance.

The above "interaction measures" may be useful in some applications as an indicator of possible design or operating problems, but they are not good measures of closed loop interaction.

It is further shown that the closed-loop *interaction* transmittance ($r_j \rightarrow y_i$) for an $m \times m$ system can be expanded into a sum of 1, 2 . . . ($m - 1$)th order terms and that in most process systems the first-order terms are dominant. This expansion provides a basis for estimating the true *closed-loop* system interactions based on the open-loop transfer function matrix (TFM). In many cases these interactions are obvious from inspection of the TFM and can be calculated using the direct Nyquist array (DNA) approach. It is demonstrated by example that the DNA gives more information about variable pairing and controller design than most of the published interaction measures.

Various numeric interaction measures can be formulated based on the elements of the transfer function matrix or the DNA. However, it is emphasized that the DNA is one part of a complete *multivariable* control system design approach and that *multiloop* systems are simply a special case that arises when the interactions are small or when they are suppressed by an appropriate compensator. The DNA can be used for interaction analysis and then the control system design completed using classical methods. However, it is recommended that a complete, consistent multivariable design method such as DNA be used since interaction analysis, pairing of input/output variables and interaction compensation are all handled directly.

Discussion of Interactions

Definition

In a multivariable system one output is in general influenced by more than one input, or conversely one reference input (set point) will influence more than one output. This is normally the situation in both open- and closed-loop systems. By extending MacFarlane's (1972) definition the term interaction can thus be logically defined:

> Interactions in a closed loop multivariable system are determined by the transmittances influencing the way in which a reference input, $r_i(s)$, or a disturbance input, $\xi_i(s)$, affects the set of outputs $\{y_j(s): j \neq i\}$, or alternatively the transmittances influencing the way in which an output $y_i(s)$ is affected by the set of reference inputs $\{r_j(s): j \neq i\}$ or disturbance inputs $\{\xi_j(s)\}$

Note that this definition means that any transmittance between a reference input, $r_i(s)$, and the corresponding output, $y_i(s)$, is not an interaction transmittance. Interactions in multivariable processes can be analyzed directly using the open-loop transfer function matrices as illustrated later. Analysis of the interactions in closed-loop feedback systems such as those shown in Figures 1 and 2 is more difficult but also more relevant to the actual performance of the system.

Closed-loop transmittances

The closed loop relationship between the output vector $y(s)$, and the reference input vector $r(s)$ and the disturbance vector $\xi(s)$ for the system in Figure 1 is

$$\begin{aligned} y &= (I + G_pK_1K_2H)^{-1}G_pK_1K_2r + (I + G_pK_1K_2H)^{-1}G_L\xi \\ &= (I + QK_2H)^{-1}QK_2r + (I + QK_2H)^{-1}G_L\xi \\ &= F^{-1}QK_2r + F^{-1}G_L\xi = Rr + R_L\xi \qquad (1) \end{aligned}$$

The matrix $K_1(s)$ is an optional precompensator matrix which can be designed (e.g., using the DNA approach) to reduce open

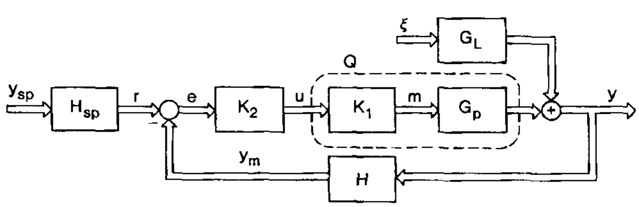

Figure 1. Block diagram of a closed loop multivariable system.

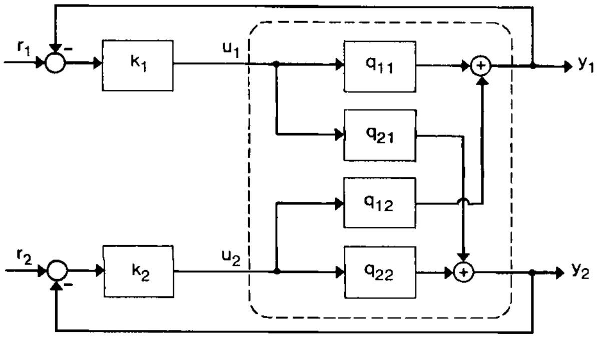

Figure 2. Block diagram of a 2×2 multiloop system.

loop interactions. If not required, then $K_1 = I$ and $Q(s)$ can be interpreted as the process transfer function matrix. K_2 is a matrix of dynamic feedback controllers, but in general the individual controller gains, k_i, are unknown at the start of the design procedure. In the above equation and those following the Laplace argument is omitted for convenience. From Eq. 1 the elements $r_{ij}(s)$ of the closed-loop transfer function matrix $R(s)$ relating y_i and r_j are given by

$$r_{ij} = \frac{\sum_{\ell=1}^{m} c_{\ell i} k_j a_{\ell j}}{\det F} \tag{2}$$

A similar expression can be stated for the elements of $R_L(s)$.

The ith row of the closed-loop transfer function matrix $R(s)$ can be viewed as a single-loop system with reference input $r_i(s)$, output $y_i(s)$, known inputs $\{r_j(s), j = 1, \ldots, m, j \neq 1\}$, and disturbance $\{\xi_k(s), k = 1, \ldots, p\}$. Furthermore it is possible to express the diagonal element $r_{ii}(s)$ of $R(s)$ as a sum of two terms: one involving only $k_i(s)q_{ii}(s)/\det F(s)$, and one involving products of cofactors of the ith column of $F(s)$ and elements of the ith column of $Q(s)$. This leads to the following remark:

Remark 1. Transmittances in a closed-loop multivariable system can be classified as direct, parallel, interaction, and disturbance transmittances, as shown schematically in Figure 3 and defined mathematically by

$$y_i = \frac{k_i q_{ii}}{\det F} r_i + \sum_{k=1}^{m} \frac{(c_{ki} - \delta_{ki}) k_i q_{ki}}{\det F} r_i + \sum_{\substack{j=1 \\ j\neq i}}^{m} \sum_{k=1}^{m} \frac{c_{ki} k_j q_{kj}}{\det F} r_j + \sum_{j=1}^{p} \sum_{k=1}^{m} \frac{c_{ki} g^L_{kj}}{\det F} \xi_j \tag{3}$$

The above equation shows that parallel, interaction, and disturbance transmittances are all dependent on the properties of the return difference matrix $F(s)$ through the cofactors $c_{ij}(s)$ of this matrix. The determinant of $F(s)$, $\det F(s)$, is a common factor to all transmittances and thus need not be considered when evaluating the *relative* magnitudes of the four transmittances. The elements of $R(s)$ for a 2×2 system as shown in Figures 1 and 2 are:

$$\begin{aligned} r_{11} &= (k_1 q_{11} + \det QK_2)/\det F \\ r_{21} &= k_1 q_{21}/\det F \\ r_{12} &= k_2 q_{12}/\det F \\ r_{22} &= (k_2 q_{22} + \det QK_2)/\det F \end{aligned} \tag{4}$$

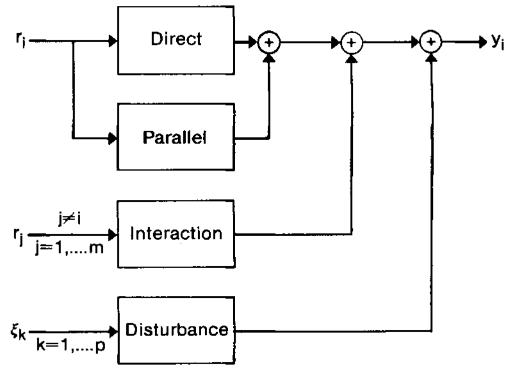

Figure 3. Block diagram showing direct, parallel, interaction, and disturbance transmittances in a multivariable system (cf. Eq. 3).

Scaling of I/O variables

In a 2×2 system the relative magnitude of the direct plus parallel transmittance $r_1 \rightarrow y_1$ vs. the interaction $r_2 \rightarrow y_1$ can be seen by comparing r_{11} vs. r_{12}. This comparison is easier to make if the I/O variables are expressed in normalized perturbation form, e.g., $y = (y - y^s)/y^{ss}$ where y^{ss} is the normal steady state value. The magnitudes of the transmittances are also a function of frequency, but as shown later this is conveniently handled by comparing elements of a DNA array.

Limiting values of transmittances

From Eq. 1 it can be seen that as the feedback controller gains $\{k_i(s), i = 1 \ldots m\}$ of $K_2(s)$ approach infinity, $R(s)$ approaches $H^{-1}(s)$ and $R_L(s)$ approaches zero. In most multiloop feedback systems $H(s)$ is diagonal and the set points, y_{sp}, are specified in the same units as the outputs y. Thus $H_{sp} = H$ in Figure 1, which leads to the following well-known result:

Remark 2. Interaction and disturbance transmittances (as defined by Eq. 3 and Figures 1 and 3) approach zero as all feedback controller gains approach infinity. Similarly the transmittances between the set points y_i^{sp} and the outputs y_i approach unity.

Closed-loop transmittances as a function of Q(s)

Equation 3 expresses the direct, parallel, interaction and disturbance transmittances as a function of the cofactors of the return difference matrix $F(s)$. As the first step in a procedure to estimate these transmittances based on open loop data, Jensen (1981) developed the following expression for the elements of $R(s)$ as a function of the minors of the *open-loop* transfer function matrix $P(s) = Q(s)K_2(s)$:

$$r_{ij} = \frac{1}{\det F}\left[P^i_j + \sum_{\ell_1=1}^{m} P^{i\ell_1}_{j\ell_1} + \sum_{\ell_1=1}^{m} \sum_{\ell_2=\ell_1+1}^{m} P^{i\ell_1\ell_2}_{j\ell_1\ell_2} + \cdots \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_m=\ell_{m-1}+1}^{m} P^{i\ell_1\ell_2\cdots\ell_m}_{j\ell_1\ell_2\cdots\ell_m} + \delta_{ij}\det P\right] \tag{5}$$

where $\ell_1, \ell_2 \ldots \ell_m$ are always different from i, j and the notation P^i_j denotes a minor of $P(s)$ formed by deleting all rows except the row(s) indicated by the superscript(s), i, and deleting all columns except those indicated by the subscript(s), j. In Eq. 5 it is assumed, without loss of generality, that $H = I$.

The transmittances for a 2×2 system are given in Eq. 4; for a 3×3 system the elements of $R(s)$ have the form:

$$r_{12} = \frac{k_2 q_{12} + k_2 k_3 (q_{12} q_{33} - q_{32} q_{13})}{\det F} \tag{6}$$

A careful examination of Eqs. 5 and 6 shows for example that for a 3×3 system the transmittances, $r_{i,j}$ consist of first- and second-order minors of $P = QK_2$. More generally:

Remark 3. The transmittances in an $m \times m$ *closed-loop* system can be expressed as a sum of terms involving minors of order $1, 2 \ldots (m - 1)$, and the kth order interaction transmittances are associated with kth order minors of the *open-loop* transfer function matrix.

The general term "associated with" is used deliberately, since it is not possible to associate a physical signal transmission path with just the numerators of the elements of $R(s)$ as given in Eq. 5. The physical transmission paths are revealed by dividing the numerators by det F using long division, which shows that there exists an infinite number of paths. The primary physical transmission paths can also be established using combinatorics on a signal flow graph.

As an example, the transmittances along the *primary* physical transmission paths for a 2×2 system are:

$$\begin{aligned} r_{11} &= k_1 q_{11} - k_1 q_{21} k_2 q_{12} + \cdots \\ r_{21} &= k_1 q_{21} + \cdots \\ r_{12} &= k_2 q_{12} + \cdots \\ r_{22} &= k_2 q_{22} - k_2 q_{12} k_1 q_{21} + \cdots \end{aligned} \tag{7}$$

For a 3×3 system

$$r_{12} = k_2 q_{12} - k_2 k_3 q_{32} q_{13} + \cdots \tag{8}$$

A physical understanding of the primary transmission paths can be obtained by tracing the expressions given above for r_{ij} on the block diagram in Figure 2. The *direct* transmittance between r_1 and y_1 is $k_1 q_{11}$ and the *parallel* transmittance is $-k_1 q_{21} k_2 q_{12}$. Similarly, the interaction transmittance $r_1 \rightarrow y_2$ is $k_1 q_{21}$ and for $r_2 \rightarrow y_1$ is $k_2 q_{12}$. Note that the parallel transmittance includes the product of the interaction transmittances. Thus although in many designs the performance objectives are to maximize the sum of the direct plus parallel transmittances and to minimize the interaction transmittances, these objectives are not independent. The reason for the infinite number of terms in the parallel and interaction transmittances can be understood by noting in Figure 2 that a change in u_1 can exit loop 1 via q_{21}, pass N times around loop 2 and then return via q_{12} where $N = 1, 2, 3. \ldots$ Equation 8 shows that for a 3×3 system the first-order interaction passes directly from r_2 to y_1 via $k_2 q_{12}$ but the second-order interaction passes through elements associated with loop 3 before reaching y_1. Since the off-diagonal elements of $Q(s)$ are typically smaller than the diagonal elements, one expects that the more elements of $q_{ij} i \neq j$ that a signal passes through, the smaller the transmittance will be. Expressed more generally:

Remark 4. For most practical systems the magnitude of the minors in Eq. 5 will normally decrease as the order increases, and hence the first-order transmittance terms provide an approximation to the total transmittance.

This is obviously true for the interaction transmittances, $r_{ij} i \neq j$, in a 2×2 system as shown by Eq. 4 and in the examples presented later. For systems larger than 2×2 it is easy to construct examples that violate remark 4, but our experience indicates it is a reasonable approximation.

The significance of the above development with respect to interaction analysis can be summarized as:

Remark 5. The relative magnitude of the closed-loop *interaction* transmittances $r_{ij} i \neq j$ with respect to the *direct* transmittance can be approximated by comparing the elements q_{ij} of the open-loop transfer function matrix with the corresponding diagonal element q_{ii}.

For example, the effect of r_i on y_i vs. the effect of r_j on y_i can be approximated by comparing q_{ii} vs. q_{ij}, as illustrated by Eq. 7. In the examples presented later this comparison will be done graphically using a Nyquist array.

Transmittances with one loop open

Much of the interaction analysis literature has considered transmittances in a multivariable system in which the ith loop is open and all other loops are closed. In this situation the relationship (cf. Eq. 1) between reference input vector $r(s)$, the disturbance vector $\xi(s)$, and the output vector $y(s)$ is

$$\begin{aligned} y &= (I + QK_2HS)^{-1} QK_2 r + (I + QK_2HS)^{-1} G_L \xi \\ &= \overline{R} r + \overline{R}_L \xi \end{aligned} \tag{9}$$

where S is a diagonal matrix with the ith diagonal element zero and all other diagonal elements unity, corresponding to all loops closed except the ith loop. An expression similar to Eq. 3, for the ith output when all loops except the ith loop are closed, can be stated as follows

$$y_i = q_{ii} u_i + L_i u_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} \overline{r}_{ij} r_j + \sum_{j=1}^{p} \overline{r}_{L_{ij}} \xi_j \tag{10}$$

where $u_i(s) = k_i(s) r_i(s)$ since the ith loop is open. The parallel transmittance, L_i under these conditions can be calculated by the following expression derived by Jensen (1981)

$$L_i = \sum_{\substack{j=1 \\ j \neq i}}^{m} \phi_{ij} q_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^{m} \phi_{ji} q_{ji} \tag{11}$$

As discussed later, the direct plus parallel transmittance, $h_i(s) = q_{ii}(s) + L_i(s)$, as the gains in all closed loops approach infinity plays an important part in several published measures of interaction. In the limit as the gain of all closed loops approaches infinity the ratio of cofactors $\phi_{ij}(s)$ of $R(s)$ approaches the corresponding ratio of cofactors of the compensated plant transfer function matrix $Q(s)$. This means that the parallel transmittance, $L_i(s)$, does not in general approach zero under high gain feedback in all closed loops, and hence should not be used as a measure of true interaction.

The Direct Nyquist Array

Interaction analysis is an inherent part of multivariable, frequency-domain design techniques such as the direct Nyquist array (DNA) method. Space precludes a complete description of the DNA method but the following are the key points relative to interaction analysis for systems such as those shown in Figures 1 and 2.

1. The Nyquist array is simply a set of standard polar plots of the magnitude and phase angle for each element of $G_p(i\omega)$ as illustrated by Figure 7. The computer-generated plots are incomplete because they are plotted only for a finite range of user specified frequencies. Only a small number of frequencies are used to save computer time and hence the loci appear piecewise continuous. Labeling of the frequencies in Figure 7 has been removed for clarity. If $K_1 = I$ then $Q = G_p$.

The DNA can be interpreted as a graphical representation of the open-loop relationship $y = Qu$. Obviously in Figure 7: the transmittances $u_1 \rightarrow y_1$, $u_2 \rightarrow y_2$ are about equal; the interaction $u_1 \rightarrow y_2$ is very significant; the interaction $u_2 \rightarrow y_1$ is negligible; and a different pairing of I/O variables (i.e., interchanging columns in Q) is not advisable. Applying the logic that leads to remark 5, it follows that for the closed-loop system r_1 will have a strong effect on y_1 and y_2 (since q_{11} and q_{21} are similar in Figure 7) but r_2 is effectively decoupled from y_1 (since q_{12} is negligible).

2. Figure 4 shows the DNA for an unspecified matrix with the Gershgorin bands superimposed on the Nyquist plots of the diagonal elements q_{11} and q_{22}. The Gershgorin bands are formed by drawing circles of radius $\Sigma_{j=1}^{m} |q_{ji}|\ i \neq j$ with centers on the q_{ii} locus. The radius is equal to the sum of the magnitudes of all the off-diagonal elements in column i. Thus comparing the magnitude of q_{ii} at a given frequency with the radius of the Gershgorin circle gives a measure of the first-order interactions vs. direct transmittances. For a 2×2 system comparing $|q_{ii}|$ vs. the radius is the same as comparing $|q_{ii}|$ vs. $|q_{ji}|$. A similar figure can be drawn using the rows of Q rather than the columns. This measure of interaction is exact for the open-loop system and an approximation for the closed-loop system.

Remark 6. The DNA display of $Q(s)$ provides a direct means of comparing the direct and interaction transmittances in the open-loop system as well as the direct and first-order interaction transmittances in the closed-loop system. This is done by graphically comparing the Nyquist plots for $q_{ij} i \neq j$ with the corresponding diagonal element, q_{ii}.

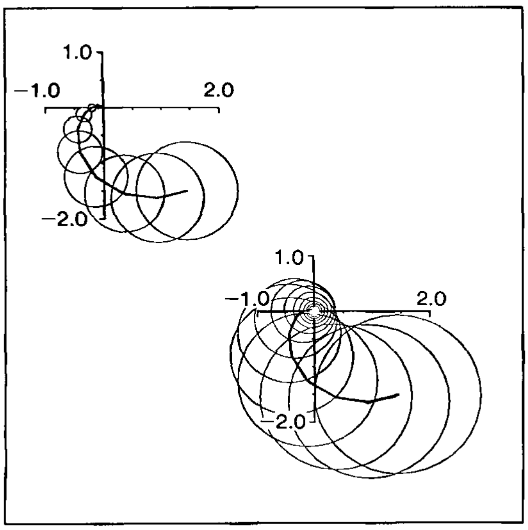

Figure 4. Diagonal elements of a Nyquist array with Gershgorin bands superimposed.

In Figure 4 the interactions are moderate with respect to q_{11} but for q_{22} the interactions are greater than the direct transmittance at higher frequencies. (This is obvious since the Gershgorin circles enclose the origin.) Economou and Morari developed an interaction measure based on their internal model control (IMC) approach to control system design and they discuss a number of examples. Despite the totally different method of derivation, their interaction measure is essentially a normalized version of the Gershgorin band, i.e., the Gershgorin radius and the magnitude of the elements of $Q(s)$ are normalized by division by $\Sigma_{j=1}^{m} |_{q_{ij}}|$, which is the sum of the magnitudes of all elements in a given row. A similar relationship is derived for column dominance. The interaction measure is plotted in a Bode rather than a Nyquist format but the end result is the same. Their conclusions can also be derived by extending this discussion of frequency domain methods such as DNA.

3. If the objective is a multiloop control system, i.e., $K_2(s)$ = diagonal, then it is common to talk about pairing the I/O variables. In the DNA method this is done by appropriate selection of K_1. Note that for a 2×2 system

$$K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

interchanges the columns and hence working with $Q = G_pK_1$ is equivalent to pairing $u_2 \rightarrow y_1$ and $u_1 \rightarrow y_2$. For most multiloop systems the objective is typically to minimize interactions by having the diagonal elements of $R(s)$ larger than the off-diagonal elements. This in turn implies that the largest elements of $Q(s)$ should be on the diagonal.

Remark 7. To minimize interactions, pair the manipulated and controlled variables so the largest elements of the DNA display of $Q(s)$ lie on the main diagonal and so that the Gershgorin bands are as small as possible.

This pairing procedure is consistent with Bristol's (1977) relative *dynamic* gain array approach but requires less numerical calculation.

4. As indicated by Eqs. 1, 3, and 5, an exact calculation of $R(s)$ requires knowledge of the controller gains in K_2. It was shown by Kuon (1975) that $h_i(s)$ (see discussion of Eqs. 9 and 10) is located within the Gershgorin band around q_{ii} regardless of the gains in the other loops, and that the stability of the multiloop system could be guaranteed by choosing k_i so that there was a suitable gain and/or phase margin on the Nyquist plot of $h_i(s)$.

Remark 8. Conservative estimates of the controller gains k_i, $i = 1 \ldots m$ can be obtained from the open-loop Nyquist plots of $q_{ii}\ i = 1 \ldots m$ with the Gershgorin bands superimposed using standard Nyquist design rules. Note, this uses only open-loop data, i.e., $Q(s)$.

For the system plotted in Figure 4, k_1 can be obtained by noting that the outer edge of the Gershgorin bands cuts the negative real axis at about 0.4. (The computer prints out the exact value.) Following the normal Nyquist design rules suggests $k_1 = 1.25$ for a gain margin of 2. Once the gains $k_i i = 1 \ldots m$ are known then the elements of $R(s)$ can be calculated and worst case calculations of closed-loop system performance can be made. [A more sophisticated program such as the one developed by Jensen (1981) will calculate a set of $k_i\ i = 1 \ldots m$ using the exact h_i loci such that user-specified stability margins are obtained. These k_i will in general be larger than those calculated based on the Gershgorin bands and hence r_{ii} will be closer to unity and $r_{ij} i \neq j$ will be smaller, as implied by remark 2.] Economou and Morari in their discussion of interactions based on their IMC approach give a limiting value of the loop gain which is equivalent to using the outer edge of the Gershgorin band as noted above.

5. Any *good* interaction measure will show that for the system plotted in Figure 7, interactions are significant. The next question is how to reduce undesired interactions. This is a standard part of multivariable DNA design. Using the DNA plot of $Q(s)$ in Figure 7, note that graphically subtracting column 2 from column 1 will significantly reduce q_{21} and will not change q_{11} significantly. The resulting DNA plot is much more diagonally dominant (i.e., off-diagonal elements are small with respect to the corresponding diagonal elements) and the interaction transmittances are almost eliminated without significant reduction in the direct transmittances. This result is achieved by choosing K_1 in Figure 1 as:

$$K_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tag{13}$$

The design of the control matrix, K_2, now proceeds using the compensated plant $Q = G_p K_1$. For implementation K_1 is included as part of the controller. DNA is much more powerful than indicated by this trivial compensator. Other decoupling techniques could also be used.

6. Aside from the features noted above, Nyquist diagrams provide significant information related to the performance of systems. For example, Edmunds (1979) shows how robustness and control quality can be analyzed using closed-loop Nyquist or Bode arrays.

Critical Review of Published Interaction Measures

Most published measures of interaction have shortcomings such as: (i) they are based on a system with one loop open rather than on a fully closed-loop system, and (ii) they measure parallel transmittance rather than the true closed-loop interaction.

The interaction quotient

One of the first suggested measures of process interaction was the interaction quotient proposed by Rijnsdorp (1965). Rijnsdorp considered only 2 × 2 systems with the emphasis on distillation column control. For a 2 × 2 system the interaction quotient is defined to be

$$\kappa = g_{12}g_{21}/g_{11}g_{22} \tag{14}$$

where $g_{ij}(s)$ is an element of the plant transfer function matrix $G_p(s)$. The quotient is evaluated for $s = i\omega$ and plotted as a polar plot. The interaction quotient has the properties that it is dimensionless and invariant under scaling. Rijnsdorp concludes that when the static value of κ is close to unity, interaction causes poor control. This is also the case when $\kappa(s)$ shows increasingly negative phase with frequency and a magnitude close to one. However, when $\kappa(s)$ is approximately constant and negative, good control can be achieved with a multiloop control system. Poor integrity results if $\kappa(s)$ is approximately constant and greater than one.

The relationship

$$\kappa = \frac{q_{12}q_{21}}{q_{11}q_{22}} = \lim_{k_1 \to \infty} \frac{q_{12}q_{21}}{q_{11}q_{22} + \dfrac{q_{22}}{k_1}}$$

$$= \lim_{k_1 \to \infty} \frac{\dfrac{k_1 q_{21} q_{12}}{1 + k_1 q_{11}}}{q_{22}} = \lim_{k_1 \to \infty} \frac{L_2}{q_{22}} \tag{15}$$

shows that the interaction quotient is the limiting value of the ratio of parallel transmittance to direct transmittance for a system with one loop closed and one loop open. Hence the interaction quotient is not a measure of interaction, but a measure of the relative significance of parallel transmittance vs. direct transmittance. Interaction transmittances are related to parallel transmittances (cf. Eq. 3), so the measure has some validity. However, it is clearly not a direct measure of closed loop interaction. This will be demonstrated by later examples.

The interaction quotient can be extended to $m \times m$ systems by the following definition

$$\kappa_i = \frac{\sum_{j=1, j\neq i}^{m} \psi_{ij} q_{ij}}{-q_{ii}} = \frac{\sum_{j=1, j\neq i}^{m} \psi_{ji} q_{ji}}{-q_{ii}} \tag{16}$$

The above extended definition follows from the expression for $L_i(s)$ in Eq. 11 and the fact that as k in $K_2(s) = kI$ approaches infinity the ratio of cofactors of the return difference matrix approach the corresponding ratio of cofactors of the compensated plant transfer function matrix (Jensen, 1981), i.e., with $K_2(s) = kI$ the $\lim_{k \to \infty} \phi_{ij} = \psi_{ij}$. It is easily verified that the extended interaction quotient defined by Eq. 16 reduces to Rijnsdorp's definition for a 2 × 2 system. The extended interaction quotient has the same properties as Rijnsdorp's measure; in addition, reordering of inputs or outputs corresponds to an equivalent reordering of the set $\{\kappa_i: i = 1, \ldots m\}$.

Kominek and Smith (1979) gave the first extensive dynamic interpretation of the interaction quotient. They used polar plots of $\kappa(s)$, $s = i\omega$, with a superimposed unit circle. They classified the plots as showing favorable and unfavorable interactions. From the preceding treatment it is evident that Kominek and Smith were actually judging whether or not parallel transmittance would be advantageous or disadvantageous in multiloop control system.

The relative gain array

Bristol (1966, 1968) suggested a steady state interaction measure, which is a relative gain ratio

$$\mu_{ij} = \frac{[(\Delta y_i/\Delta u_j) | \Delta u_k = 0 \quad \text{for} \quad k \neq j]}{[(\Delta y_i/\Delta u_j) | \Delta y_k = 0 \quad \text{for} \quad k \neq i]} \tag{17}$$

The numerator corresponds to the open-loop gain between u_j and y_i, and the denominator is the gain between u_j and y_i with all other outputs perfectly controlled. $M = \{\mu_{ij}\}$ is called the relative gain array (RGA). The RGA, like the interaction quotient, is restricted to plants or compensated plants with an equal number of inputs and outputs. By virtue of the defining equation the elements of the relative gain array are dimensionless. This implies they are invariant under scaling, and since selection of final proportional controller gains is a scaling operation, the relative gain array is independent of the final multiloop controller gains K_2. The RGA further has the property that any row or column sums to one, and that reordering of inputs or outputs implies a similar reordering of columns or rows of M. Bristol has shown that only the open loop gains are necessary to evaluate the relative gain array, since

$$M = Q(0) \circ [Q(0)^{-1}]^T = \{q_{ij}(0)\hat{q}_{ji}(0)\} \tag{18}$$

where $\hat{q}$ is an element of Q^{-1} and the symbol $\circ$ indicates the Hadamard or Schur product of the two matrices (Johnson, 1974). Bristol (1968) recommends that controlled and manipulated variables with μ_{ij} positive and close to unity be paired. Also if any μ_{ij} is much larger than one or less than zero, pairing the corresponding variables will result in a loop that is difficult to control. Bristol (1977) has also shown that pairings which give negative relative gains on the diagonal result from a control loop with nonminimum phase characteristics. The RGA has proven to be very effective for a number of practical applications. However, the user should be aware that the RGA can give misleading indications about the amount of interaction present in dynamic systems and even at steady state (see later discussion of triangular systems and example 1).

The steady state nature of the RGA neglects any high-frequency dynamics, which could be important in certain systems, such as the turbo alternator considered by Ahson and Nicholson (1976). Witcher and McAvoy (1977) and Bristol (1978) therefore suggested an intuitive dynamic extension by defining a relative dynamic gain array (RDGA) as follows

$$M(s) = Q(s) \circ [Q(s)^{-1}]^T = \{q_{ij}(s)\hat{q}_{ji}(s)\} \tag{19}$$

which possesses the same dimensionless properties as the steady state equivalent. Witcher and McAvoy state that interaction is small if the magnitude of the diagonal elements of $M(s)$ are close to one and the magnitude of all other elements is small. This is equivalent to saying that $M(s)$ should be a diagonally dominant matrix. The information contained in the RDGA is best seen by plotting it as an array of polar plots similar to the DNA display (as in Figures 6 and 8). A more complete discussion of the interpretation of the RDGA display is given by Jensen (1981).

The relationship, for a 2 × 2 system

$$\mu_{22} = \frac{q_{22}}{q_{22} - \dfrac{q_{21}q_{12}}{q_{11}}} = \lim_{k_1 \to \infty} \frac{q_{22}}{q_{22} - \dfrac{k_1 q_{21} q_{12}}{1 + k_1 q_{11}}}$$

$$= \lim_{k_1 \to \infty} \frac{q_{22}}{q_{22} + L_2} \tag{20}$$

shows that the RDGA elements are the limiting values of the ratio of direct transmittance to the sum of direct and parallel transmittance for a system with one loop open and the rest closed. A relationship similar to Eq. 20 for higher-order systems is easily derived.

Therefore, the RDGA does not provide a direct measure of the true closed loop interactions. However, it works in many applications because (from Eq. 20) $\mu_{22} \to 1$ implies that $L_2 \to 0$, and as seen from Eq. 7 this means that at least one interaction transmittance must be small. But consider a multivariable system where one of the q_{ii} is small. It may then be desirable to tolerate some interaction so the parallel transmittance from r_i to y_i will be significant. Thus μ_{ii} = small might be better than $\mu_{ii} \to 1$. (Cases such as this must be considered on an individual basis.) Also, any system with a triangular transfer function matrix (TFM), $G(s)$, has a RDGA equal to the identity matrix, which appears to be the "ideal" case. (For a 2 × 2 system such as in Figure 2 the closed loop TFM, $R(s)$, is also triangular.). However, there can still be significant one-way interaction, as shown later in example 1. McAvoy (1983a) recommends the use of the RDGA and presents five case studies to illustrate its usefulness. Jensen (1985) in a note to the editor strongly challenges McAvoy's approach and shows that simple inspection of the open loop transfer function matrix, $G(s)$, (or use of the DNA) is a simpler and better way of analyzing interactions.

The extended interaction quotient (cf. Eq. 16) and the diagonal elements of the relative dynamic gain array are related as follows (Jensen, 1981):

$$\mu_{ii} = \frac{1}{1 - \kappa_i} \quad \text{or} \quad \kappa_i = \frac{\mu_{ii} - 1}{\mu_{ii}} \tag{21}$$

Tung and Edgar's RDGA

Tung and Edgar (1977, 1978, 1981) use both state space and frequency domain approaches to develop a relative dynamic gain array measure of interaction that reduces to Bristol's (1966) RGA at steady state. In fact their approach includes a rigorous mathematical analysis that illustrates the basis for Bristol's RGA which he originally derived intuitively. In addition to reducing interactions they also consider the closed-loop eigenvalues that can be achieved with multiloop control using different I/O variable pairings. In case 2 (1977) they conclude that the pairing indicated by their RDGA was capable of moving the closed-loop poles much farther to the left in the complex plane than was possible using the pairing indicated by a steady state interaction analysis. In some respects their dynamic approach is a simplified and less restrictive formulation of the simultaneous decoupling and pole placement problem. However, the disadvantages are that they require a state space model and they assume a statically decoupled system before proceeding with their dynamic interaction and stability analysis. The reason for assuming a statically decoupled model (equivalent to $Q(s) = G_p(s)G(0)^{-1}$) is not explained and could lead to difficulties in some special cases. The authors explicitly point out (1977) that loops that are judged noninteractive through their analysis are not necessarily decoupled in the closed-loop sense. The DNA approach combined with time domain simulation appears to be a more versatile and easy to understand approach.

The average dynamic gain array

Gagnepain and Seborg (1979) use open-loop step responses as the basis for pairing variables. They extend the dynamic potential idea of Witcher and McAvoy (1977), using integrals of open-loop step responses to define an average dynamic gain array. The average dynamic gain array (ADGA) is defined by

$$M(\theta) = D(\theta) \circ [D(\theta)^{-1}]^T \tag{22}$$

where each element of $D(\theta)$ is an integral of an open-loop step response from time θ_1 to time θ_2 divided by the integration interval. θ_1 is chosen as the smallest time at which the matrix $D(\theta)$ is not singular if integration was from time zero to time θ_1, and θ_2 is chosen so the integration interval $\theta = \theta_2 - \theta_1$ is equal to the largest time constant of the plant transfer function matrix. The definition of the ADGA is in complete analogy with the definition of the RDGA, and it therfore has many of the properties of the RDGA. In particular it provides answers to the same question as the RDGA, i.e., since $D(\theta)$ is in effect a matrix of average open loop gains, the ADGA measures the significance of parallel vs. direct transmittance in the same way and for the same situation as the RDGA. The difference between the RDGA and the ADGA is that the latter tries to condense the dynamic information into a single matrix of numbers. Gagnepain and Seborg show that the ADGA, even though it has a higher success rate than the steady state RGA, fails in the crucial situation where stability considerations determine the best pairing of variables. This confirms Tung and Edgar's findings that pairing for minimum interaction is not always the best approach.

The relative transient response functions

Jaaksoo (1979) presented an exact time domain equivalent of the RDGA. Based on a discrete state space model of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{23}$$

relative transient response functions $\phi_{ij}(k)$ are calculated. These functions are defined by

$$\phi_{ij}(k) = \frac{\sum_{t=1}^{k} c_i A^{t-1} b_j}{\sum_{t=1}^{k} c_i (M_{ij}A)^{t-1} M_{ij} b_j} = \frac{c_i \left[\sum_{t=1}^{k} A^{t-1} \right] b_j}{c_i \left[\sum_{t=1}^{k} (M_{ij}A)^{t-1} \right] M_{ij} b_j} \tag{24}$$

Due to the exact equivalence with Bristol's (1966) approach, the relative transient response functions give a measure of the significance of parallel vs. direct transmittance with one loop open and the rest closed, and not a direct indication of closed-loop interaction. Jaaksoo (1979) suggests using only the arrays obtained with $k = 1$ and $k \rightarrow \infty$ in the analysis. This corresponds to using the RDGA with only $s \rightarrow \infty$ and $s = 0$, respectively. Jaaksoo introduces this limitation because of difficulties in the analytical investigation of a set of time functions. For example the matrix power series in Eq. 24 will only converge if all eigenvalues of A and $[M_{ij}A]$ have absolute values less than one, and the denominators can only be calculated from the numerators for $k = 1$. Further drawbacks of the relative transient response functions are: it is not clear how the functions should be displayed or interpreted as functions of the time parameter k; they are not directly associated with a multivariable control system design technique; and they require a significant number of numerical calculations to evaluate.

Measures of interaction that use the control structure

None of the measures or indices discussed so far requires any knowledge of the control system structure, the type of final controllers, or the associated gains. The result is that the indices reviewed are more of a warning about potential difficulties than they are indicators of closed-loop interaction. Three measures, which require the control system structure and the final controller to be known, will now be reviewed.

Davison (1969) suggests a nonminimum phase index and an interaction index based on a state space model. The nonminimum phase index is evaluated from eigenvalues of two matrices for each input-output pair, so the variable pairing must be known, and the only type of controller considered is proportional feedback. The interaction index is the difference between the closed-loop nonminimum phase index and the open-loop nonminimum phase index. The nonminimum phase index is really a measure of the importance of the process dead time relative to the dominant time constant, and as such has little to do with interaction.

Davison and Man (1970) suggest an interaction index based on the difference between a single closed-loop response and the multiloop closed-loop response. The index is based on a state space model and is calculated by solving two matrix equations iteratively and calculating the maximum eigenvalues of the resulting matrices, all for each closed loop. Even though the index actually measures interaction in the closed-loop system, the amount of numerical calculation involved seems prohibitive for use in control system design.

Suchanti and Fournier (1973) suggest the interaction coefficient defined for each loop as

$$I_j = [(\overline{IE})_j - (IE)_j]/(IE)_j \tag{25}$$

where $(\overline{IE})_j$ is the error integral of the jth output for a unit step change in the jth input with the jth loop closed and all other loops open. Similarly, $(IE)_j$ is the error integral of the jth output for a unit step change in all inputs with all loops closed. All integrals are evaluated with proportional plus integral controllers tuned using single-loop techniques, i.e., neglecting the interaction. There is little doubt that the index gives a measure of closed-loop interaction, and it is possible to evaluate the coefficients for any control system. However, again the amount of numerical calculation seems prohibitive for use in control system design.

Comparison of DNA Vs. Other Interaction Measures

In this section the type of results obtained from the use of measures of interaction, such as the interaction coefficient of Rijnsdorp, the RGA, and RDGA, and the ADGA, are compared with the information obtained from the DNA.

Example 1

A simple 2 × 2 transfer function matrix will show that interaction transmittances can be severe even when parallel transmittance with one loop open is small. Consider the plant

$$G_p(s) = \begin{bmatrix} \dfrac{1}{1s+1} & \dfrac{0.05}{10s+1} \\ \dfrac{1}{2s+1} & \dfrac{1}{1s+1} \end{bmatrix} \tag{26}$$

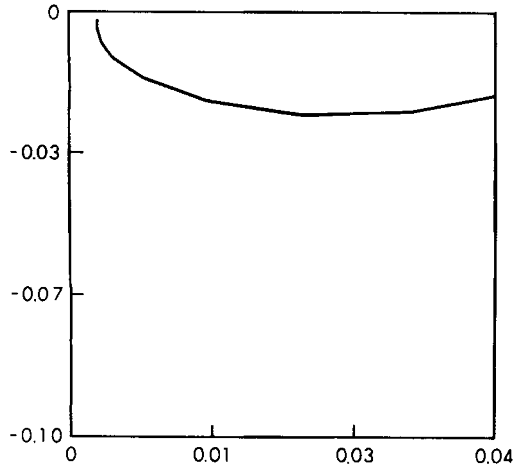

Figure 5. Rijnsdorp's interaction quotient for example 1.

Then the interaction quotient $\kappa(s)$ of Rijnsdorp and the (2, 2) element of Bristol's RDGA are respectively

$$\kappa = 0.05 \frac{1 + 2s + 1s^2}{1 + 12s + 20s^2}$$

$$\text{and} \quad \mu_{22} = \frac{1 + 12s + 20s^2}{0.95 + 11.9s + 19.95s^2} \quad (27)$$

κ and the RDGA $M(s)$ are plotted as functions of frequency $s = i\omega$ in Figures 5 and 6 respectively. The magnitude of $\kappa(s)$ is small at all frequencies, and the magnitude of the diagonal elements of $M(s)$ are close to one at all frequencies. Thus both measures in their original interpretation indicate little interaction, which really should be interpreted as meaning no difficulty in designing a multiloop control system.

The true closed loop response of the second output with two proportional controllers, expressed as the sum of direct, parallel,

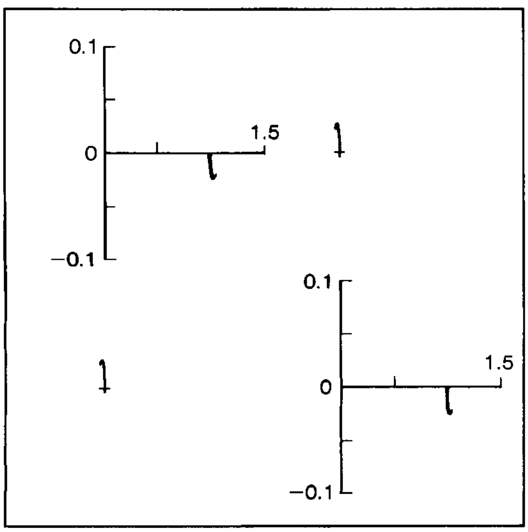

Figure 6. Bristol's RDGA (relative dynamic gain array) for example 1.

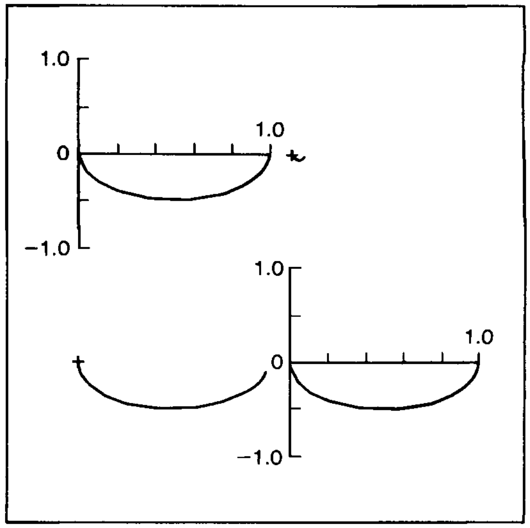

Figure 7. DNA (direct Nyquist array) for example 1.

and interaction transmittances is:

$$y_2 = \frac{1}{\det F}\left[k_2 \frac{1}{s+1} r_2 + k_2 k_1 \cdot \left(\frac{1}{(s+1)^2} - \frac{0.05}{1 + 12s + 20s^2}\right) r_2 + k_1 \frac{1}{2s+1} r_1\right] \quad (28)$$

It is evident that direct transmittance and interaction transmittance have very similar dynamics and magnitude. Further, the parallel transmittance is helpful and significant, especially at low frequencies.

With the second loop open the response of the second output becomes

$$y_2 = k_2 \frac{1}{s+1} r_2 - k_2 \frac{0.05(s+1)}{(2s+1)(10s+1)(s+1+k_1)} r_2 + k_1 \frac{(s+1)}{(2s+1)(s+1+k_1)} r_1 \quad (29)$$

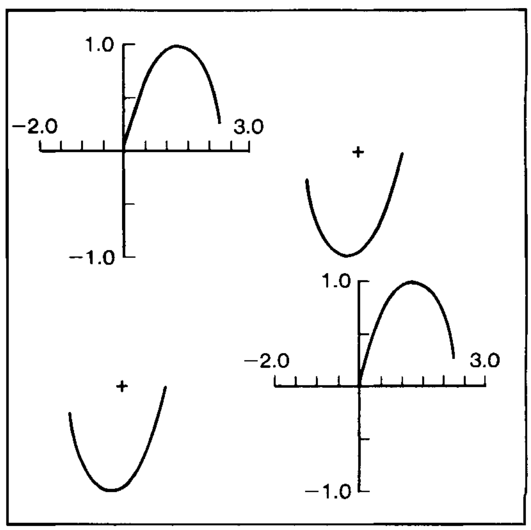

Figure 8. Bristol's RDGA for example 2.

which shows insignificant parallel transmittance under these conditions, but comparable direct and interaction transmittances. Thus interaction measures such as the RDGA that measure parallel transmittance (with one loop open) can give misleading results.

Rijnsdorp's interaction quotient and the RDGA suggest that a multiloop control system for this example can be designed without difficulty, but they fail to give a reliable measure of the closed loop interaction. For this particular plant the DNA of $G_p(s)$ gives the same information for multiloop design as the RDGA. In addition the DNA, as shown in Figure 7, indicates correctly that there is significant closed-loop interaction in the second loop, but almost none in the first. This would be even clearer if the Gershgorin bands were superimposed on Figure 7 as they are in Figure 4.

Example 2

Gagnepain and Seborg (1979) consider the following system

$$G_p(s) = \exp(-s)\begin{bmatrix} \dfrac{2}{10s+1} & \dfrac{1.5}{s+1} \\ \dfrac{1.5}{s+1} & \dfrac{2}{10s+1} \end{bmatrix} \tag{30}$$

which has the following RGA and ADGA respectively

$$\begin{bmatrix} 2.29 & -1.29 \\ -1.29 & 2.20 \end{bmatrix} \text{ and } \begin{bmatrix} -0.42 & 1.42 \\ 1.42 & -0.42 \end{bmatrix} \tag{31}$$

The two measures recommend different pairings. Gagnepain and Seborg show that the pairing recommended by the ADGA is structurally monotonic unstable according to a theorem due to Niederlinski (1972). The failure of the ADGA is due to the averaging of the integration process. The DNA plot in Figure 9 supplies the same information for pairing as the RDGA in Figure 8, and also provides a basis for designing the appropriate controller.

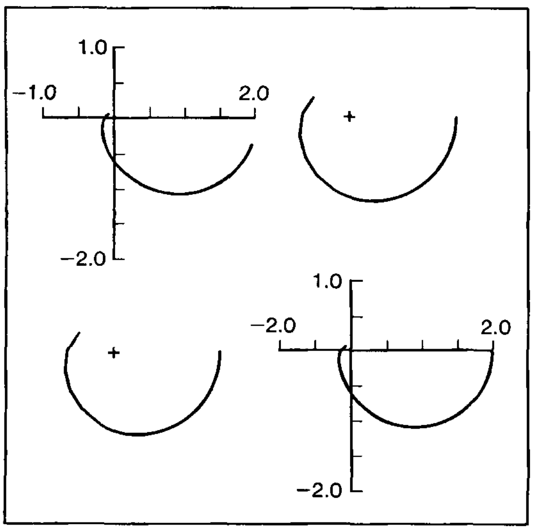

Figure 9. DNA for example 2.

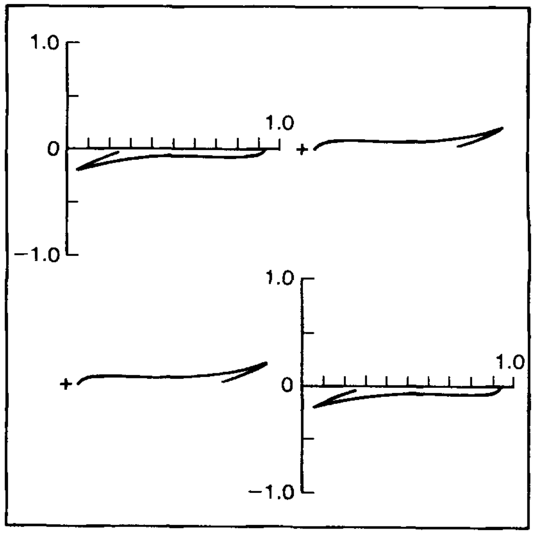

Figure 10. RDGA for example 3.

Example 3

Ahson and Nicholson (1976) consider the design of a compensator/controller for a turbo-alternator model with two inputs and two outputs. Simulations by Ahson and Nicholson show that a multiloop control system gives very unsatisfactory control. However, the steady state RGA is

$$\begin{bmatrix} 1.0235 & -0.0235 \\ -0.0235 & 1.0235 \end{bmatrix} \tag{32}$$

which indicates that there should be no difficulties in designing a multiloop control system. Note that for this system the magnitude of the elements of the RDGA change significantly with frequency, as seen from Figure 10, indicating difficulties at higher frequencies. The same information is also evident from the DNA plot in Figure 11, which also shows that the difficulties are due to the small size of the (1, 1)-element of the plant transfer function matrix. Hence a full, dynamic compensator, $K_1(s)$ is

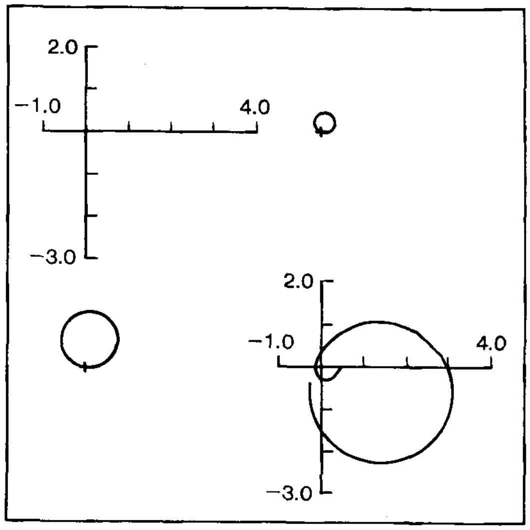

Figure 11. DNA for example 3.

probably required for effective control. The complete design of such a compensator is not relevant here, but note that the classification of transmittances in Figure 3 provides a convenient means of comparing some of the design alternatives:

1. Introduction of a diagonal precompensator to multiply the first column of $G_p(s)$ by a constant, N_1, increases the direct, parallel, and $r_1 \rightarrow y_2$ interaction transmittance by N_1.

2. Introduction of a diagonal postcompensator to multiply the first row of $G_p(s)$ by a constant, N_2, increases the direct, parallel, and $r_2 \rightarrow y_1$ interaction transmittance by N_2.

3. For a given increase in direct transmittance it is probably best to combine the above two approaches, which increases the direct transmittance by $N_1 * N_2$, the parallel transmittance by $N_1 * N_2$, the $r_1 \rightarrow y_2$ interaction by N_1, and the $r_2 \rightarrow y_1$ interaction by N_2. Additional design steps could reduce both interactions further. Also, the diagonal postcompensator can be implemented as part of the controller and hence causes no practical difficulties. The DNA thus provides a better basis for analysis and design of a compensator than the RDGA.

When a full dynamic model of the process is available, then it is recommended that a complete analysis and design be completed using a method such as DNA. When only steady state gains are available Bristol's (1966) RGA *may* give useful information. However, the user should be aware that the RGA can fail without warning.

Acknowledgment

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Notation

A = $n \times n$ state map of discrete state space model
B = $n \times m$ input map of discrete state space model
B_j = B without jth column
b_j = jth column of B
C = $m \times n$ output map of discrete state space model
C_i = C without ith row
c_i = ith row of C
$c_{ij}(s)$ = (i, j)th cofactor of $F(s)$
D = $m \times m$ matrix of average open-loop gains
$\det F$ = determinant of $F(s)$
$F(s)$ = return difference matrix
$G_L(s) = \{g_{ij}^L(s)\}$ = $m \times p$ load transfer function matrix
$G_p(s) = \{g_{ij}(s)\}$ = $m \times m$ plant transfer function matrix
$H(s)$ = $m \times m$ feedback transfer function matrix
I = identity matrix
I_j = interaction coefficient of Suchanti and Fournier (1973)
$(\overline{IE})_j$ = error integral of jth output for unit step change in jth input with jth loop closed and all others open
$(IE)_j$ = error integral of jth output for unit step change in all input with all loops closed
$K_1(s)$ = $m \times m$ compensator transfer function matrix
$K_2(s) = \mathrm{diag}\{k_i(s)\}$ = $m \times m$ controller transfer function matrix
$L_i(s)$ = parallel transmittance in system with one loop open
$M = \{\mu_{ij}\}$ = relative gain array (RGA)
$M(s) = \{\mu_{ij}(s)\}$ = relative dynamic gain array (RDGA)
$M = \{M_{ij}\}$ = $I + B_j(C_iB_j)^{-1}C_i$ in Eq. 24
N_1 = constant in diagonal precompensator, diag $\{N, 1\}$, used in example 3
N_2 = constant in diagonal postcompensator, diag $\{N, 1\}$, used in example 3
$Q(s) = \{q_{ij}(s)\}$ = $G(s)K_1(s)$
$Q^{-1}(s) = \{\hat{q}_{ij}(s)\}$ = inverse of $Q(s)$
Q_j^{ii} = minor of $Q(s)K_2(s)$ formed by deleting all rows except the ith and all columns except the jth
$R(s) = \{r_{ij}(s)\}$ = closed-loop plant transfer function matrix
$R_L(s) = \{r_{ij}^L(s)\}$ = closed-loop load transfer function matrix
$\overline{R}(s) = \{\overline{r}_{ij}(s)\}$ = plant transfer function matrix with all but one loop closed
$\overline{R}_L(s) = \{\overline{r}_{Lij}(s)\}$ = load transfer function matrix with all but one loop closed
$r(s) = \{r_i(s)\}$ = reference input vector
S = diagonal matrix with ith diagonal element zero and all other diagonal elements unity
$u_i(s)$ = manipulated variable
$u(k)$ = vector of manipulated variables in discrete state space model
$x(k)$ = state vector in discrete state space model
$y(s) = \{y_i(s)\}$ = output vector
$y(k)$ = output vector in discrete state space model
$\kappa(s)$ = interaction quotient as defined by Rijnsdorp (1965)
δ_{ij} = Kronector delta ($\delta_{ii} = 1, \delta_{ij} = 0$)
$\kappa_i(s)$ = extended interaction quotient
$\phi_{ij}(s)$ = ratio of (i, j)th to (i, i)th cofactor of $F(s)$
$\psi_{ij}(s)$ = ratio of (i, j)th to (i, i)th cofactor of $Q(s)$
$\phi_{ij}(k)$ = relative transient response function between input j and output i
$\xi(s) = \{\xi_i(s)\}$ = disturbance vector

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